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Quantum Feedback Networks: Hamiltonian Formulation

J. Gough¹, M.R. James²

Abstract

A quantum network is an open system consisting of several component Markovian input-output subsystems interconnected by boson field channels carrying quantum stochastic signals. Generalizing the work of Chebotarev and Gregoratti, we formulate the model description by prescribing a candidate Hamiltonian for the network including details the component systems, the field channels, their interconnections, interactions and any time delays arising from the geometry of the network. (We show that the candidate is a symmetric operator and proceed modulo the proof of self-adjointness.) The model is non-Markovian for finite time delays, but in the limit where these delays vanish we recover a Markov model and thereby deduce the rules for introducing feedback into arbitrary quantum networks. The type of feedback considered includes that mediated by the use of beam splitters. We are therefore able to give a system-theoretic approach to introducing connections between quantum mechanical state-based input-output systems, and give a unifying treatment using non-commutative fractional linear, or Möbius, transformations.

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1 Introduction

The aim of this paper is to contribute to our understanding of quantum feed-forward and feedback networks by introducing algebraic rules describing how to obtain an effective model for a network starting from the canonical description of the component devices as unconnected systems and the prescription of the interconnections between these systems. In the open systems approach to quantum mechanics, a unitary dynamics is given for a quantum mechanical system (e.g. atom, optical cavity, quantum dot, etc.) and its environment (e.g. optical field). When the auto-correlation time of the environment processes is small we can employ quantum stochastic approximations and work with a quantum stochastic unitary evolution [18] with associated Heisenberg-Langevin equations

of motion [13]. The appropriate way to think of the open system is as an input-output system [12] where the input process is a causal field representing the environment and the output is the scattered field after interaction with the system. Cascading such systems is a basic example of feedforward [7][11]. More generally we can consider feedback connections [21][4][6][14][15][20] [19].

The natural generalization of this is to consider a graph with quantum fields propagating along the edges and quantum mechanical systems at the vertices. The simplest quantum network is a single system with input and output, and for Markov models the evolution is described in terms of a Hudson-Parthasarathy quantum stochastic unitary adapted process [18] with the system space as initial space and the inputs as noise. There is an alternative description based on the Chebotarev-Gregoratti Hamiltonian model [9][17]. The simplest nontrivial quantum network will then be two components cascaded in series, and for finite time delays this will be non-Markovian: there are several technical and conceptual difficulties in modelling this in terms of standard (quantum) stochastic calculus. To resolve these issues, we extend the Chebotarev-Gregoratti Hamiltonian model to networks, incorporating the various interconnections and time delays into the boundary conditions that define the domain of the Hamiltonian. It is the authors' opinion that this offers the only feasible way to address the topologically non-trivial situation of quantum feedback induced using beam-splitter devices. We will show that the limit of small time delays leads to a Chebotarev-Gregoratti Hamiltonian of the type associated with Markovian models. From this we deduce the rules for introducing feedback/feedforward connections into assemblies of Markovian components starting from the component model description. The natural mathematical language to describe this is in terms of non-commutative fractional linear transformations of the type introduced by C.L. Siegel.

The notion of quantum feedback for quantum input-output systems has been around in one form or another since the late 1970's and has had a major influence on theoretical physics and engineering considerations relating to the rapidly developing field of quantum feedback control. Our results give a system-theoretic approach to introducing feedback. An important step towards a general theory of feedforward and feedback connections was made in the papers of Gardiner [11] and Carmichael [7] who considered quantum optical networks consisting of cascade-connected components with no gauge couplings, and Yanagisawa and Kimura [22], [23] who studied the situation where the plants are multi-dimensional oscillator systems and the external inputs are Bose fields coupling to the plants via emission/absorption interactions. Yanagisawa and Kimura were able to exploit the linearity of the dynamics and apply transfer function techniques to the resulting networks. The present paper deals with general quantum dynamical systems with gauge couplings and is not restricted to linear systems. The field channels are assumed to carry quantum stochastic signals that satisfy the canonical Itô table.

In section 2 we review the quantum stochastic and Hamiltonian models for open quantum systems, and provide some structural results concerning the parameters used to define the models. Our general Hamiltonian description of

quantum networks is given in section 3. In section 4 we show how edges can be eliminated to provide simpler Markovian network models. Key examples are given in section 5, and concrete topological rules are inferred in section 6.

2 Quantum Markov Input-Output Components

The concept of quantum input-output systems originates from two independent sources: the quantum theory of filtering where the output field is the object of indirect nondemolition measurement [2][3], and the theory of quantum amplifiers [8]. The latter theory was generalized to a quantum network by Yurke and Denker [25] where the network is second quantized quantum wire model with quanta satisfying a Klein-Gordon equation with Kirchhoff boundary conditions at the vertices (though there is no quantum system associated with the vertices!). Similarly, starting from a Lagrangian formulation, Gardiner and Collett, cf. [12], developed the theory of quantum electromagnetic input-output fields interacting with a quantum mechanical system at the origin. We may consider this set up as the simplest network consisting of an input channel and an output channel meeting at a vertex (the system). In the dipole approximation they derive a Langevin equations for the canonical observables of the system. Following a rotating wave approximation, and a low frequency limit they obtain a quantum white noise theory suitable for quantum optics models and which is formally equivalent to a Hudson-Parthasarathy quantum stochastic evolution for a quantum diffusion [18]. The formulation of the boundary conditions at the vertex is then a crucial aspect of the model prescription and relates the output field to the input and the system degrees of freedom.

2.1 Quantum Stochastic Process Description

The *system* is modelled as quantum mechanical with Hilbert space \mathfrak{h} . The inputs and outputs are carried along semi-infinite *quantum channels* and are modelled as quantum field signals. Specifically, we model the field quanta as propagating along the channels with constant velocity c in the direction specified by the arrows. We may then parameterize both the input and output lines by a single geometric parameter t measuring the *arc-time* taken to reach the system. A single component is sketched below as a two port device having an input and an output port.

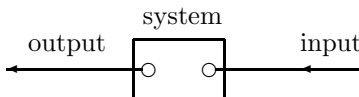


Figure 1: input-output component

The inputs correspond to the half line $\mathbb{R}^+ = (0, \infty)$ as they have yet to reach the system ($t = 0$), while the outputs correspond to $\mathbb{R}^- = (-\infty, 0)$ as they have

already passed through the system. Signals therefore have state space $L^2_{\mathfrak{K}}(\mathbb{R}, dt)$ where \mathfrak{K} is a fixed Hilbert space called the *multiplicity space*. We shall generally consider $\mathfrak{K} = \mathbb{C}^n$ which means that we have n distinguishable particles. We shall consider an indefinite number of these quanta in the wire so that the Hilbert space is the Fock space

$$\mathfrak{F} = \Gamma(L^2_{\mathfrak{K}}(\mathbb{R}, dt)),$$

where $\Gamma(\cdot)$ is the bosonic Fock space functor. Note that Fock spaces have the functorial property

$$\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \cong \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2),$$

and so $\mathfrak{F} = \mathfrak{F}_{\text{in}} \otimes \mathfrak{F}_{\text{out}}$ where $\mathfrak{F}_{\text{in}} = \Gamma(L^2_{\mathfrak{K}}(\mathbb{R}^+, dt))$ and $\mathfrak{F}_{\text{out}} = \Gamma(L^2_{\mathfrak{K}}(\mathbb{R}^-, dt))$.

2.2 Process Description of a Single Markov Component

Let us fix the multiplicity number as n and take $\{e_j : j = 1, \dots, n\}$ as a basis for $\mathfrak{K} = \mathbb{C}^n$. We denote by $A_i(t) \triangleq A(e_i \otimes 1_{[0,t]})$ and $A_i^\dagger(t) \triangleq A^\dagger(e_i \otimes 1_{[0,t]})$ the operators describing the *annihilation* and *creation* of a quantum in the i th channel over the time interval $[0, t]$, respectively. The operator describing the scattering from the j th channel to the i th channel over the time interval $[0, t]$ is denoted by $\Lambda_{ij}(t)$. In particular, $N_i(t) = \Lambda_{ii}(t)$ is the observable corresponding to the number of quanta in the i th channel over this time.

We now consider a *quantum stochastic evolution* as a unitary adapted process $\{V(t) : t \geq 0\}$ on $\mathfrak{h} \otimes \mathfrak{F}$ in the sense of Hudson and Parthasarathy [18] arising as the solution to the quantum stochastic differential equation (QSDE)

$$dV = (dG)V, \quad V(0) = 1,$$

where

$$\begin{aligned} dG(t) = & (S_{ij} - \delta_{ij}) \otimes d\Lambda_{ij}(t) + L_i \otimes dA_i^\dagger(t) \\ & - L_i^\dagger S_{ij} \otimes dA_j(t) - \left(\frac{1}{2}L_i^\dagger L_i + iH\right) \otimes dt \end{aligned} \quad (1)$$

With $S = S_{ij} \otimes |e_i\rangle\langle e_j|$ unitary on $B(\mathfrak{h} \otimes \mathfrak{K})$, $L = L_i \otimes |e_i| \in B(\mathfrak{h} \otimes \mathfrak{K}, \mathfrak{h})$, and $H \in B(\mathfrak{h})$ self-adjoint. It is convenient to write these as matrices

$$S \triangleq \begin{pmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \cdots & S_{nn} \end{pmatrix}, L \triangleq \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix}.$$

We introduce the processes

$$\begin{aligned} j_t(X) & \triangleq V^\dagger(t)[X \otimes 1]V(t), \\ \tilde{A}_i(t) & \triangleq V^\dagger(t)[1 \otimes A_i(t)]V(t). \end{aligned}$$

for $\mathbf{X} \in B(\mathfrak{h})$. They satisfy the QSDEs

$$\begin{aligned} dj_t(\mathbf{X}) &= j_t(\mathcal{L}(\mathbf{X}))dt + j_t(S_{ji}^\dagger[\mathbf{X}, \mathbf{L}_j])dA_i^\dagger + j_t([L_i^\dagger, \mathbf{X}]S_{ij})dA_j \\ &\quad + j_t(S_{ki}^\dagger(\mathbf{X} - 1)S_{kj})d\Lambda_{ij}(t), \end{aligned} \quad (2)$$

$$d\tilde{A}_i(t) = j_t(S_{ij})dA_j(t) + j_t(L_i)dt. \quad (3)$$

where $\mathcal{L}(\mathbf{X}) = \frac{1}{2}L_i^\dagger[\mathbf{X}, L_i] + \frac{1}{2}[L_i^\dagger, \mathbf{X}]L_i - i[\mathbf{X}, H]$.

2.3 System Parameters

The triple (S, L, H) , which determines the model, is referred to as the set of system parameters. The coefficients of the QSDE are assembled into the following square matrices of $(1+n)$ dimensions having entries that are operators on \mathfrak{h} :

$$\mathbf{G} \triangleq \begin{pmatrix} -\frac{1}{2}L^\dagger L - iH & -L^\dagger S \\ L & S - 1 \end{pmatrix}, \quad (4)$$

$$\mathbf{V} \triangleq \mathbf{G} + \mathbf{\Pi} = \begin{pmatrix} -\frac{1}{2}L^\dagger L - iH & -L^\dagger S \\ L & S \end{pmatrix}, \quad (5)$$

$$\mathbf{M} \triangleq \mathbf{1} + \mathbf{\Pi}\mathbf{G} = \mathbf{1} - \mathbf{\Pi} + \mathbf{\Pi}\mathbf{V} = \begin{pmatrix} 1 & 0 \\ L & S \end{pmatrix}, \quad (6)$$

where $\mathbf{\Pi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We refer to \mathbf{G} as the *Itô generator matrix* of the unitary evolution, and \mathbf{V} as the *model matrix*. The matrix \mathbf{M} is called the *Galilean transformation* associated with \mathbf{G} .

Definition 1 Let \mathfrak{h} and \mathfrak{K} be fixed Hilbert spaces. The classes of Itô generator matrices $\mathfrak{G}(\mathfrak{h}, \mathfrak{K})$ and model matrices $\mathfrak{M}(\mathfrak{h}, \mathfrak{K})$ are collections of operators $\mathbf{G}, \mathbf{V} \in B(\mathfrak{h} \otimes (\mathbb{C} \oplus \mathfrak{K}))$ of the form (4) and (5) respectively with respect to the decomposition $\mathfrak{h} \otimes (\mathbb{C} \oplus \mathfrak{K}) = \mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{K})$ for $S \in B(\mathfrak{h} \otimes \mathfrak{K})$ unitary, $L \in B(\mathfrak{h}, \mathfrak{h} \otimes \mathfrak{K})$, and $H \in B(\mathfrak{h})$ self-adjoint.

It is convenient to set $A^{00} = t, A^{i0} = A_j^\dagger, A^{0j} = A_i$ and $A^{ij} = \Lambda_{ij}$ and write $dX = X_{\alpha\beta}dA^{\alpha\beta}$ for a general stochastic integral. We adopt the convention that repeated Greek indices are summed over $0, 1, \dots, n$. The coefficients $X_{\alpha\beta}$ can be assembled into a matrix \mathbf{X} of adapted entries. We may compress the quantum Itô table down to $dA^{\alpha\beta}dA^{\mu\nu} = \hat{\delta}^{\beta\mu}dA^{\alpha\beta}$ where $\hat{\delta}^{\alpha\beta}$ is the Hudson-Evans delta which equals unity when $\alpha = \beta \in \{1, \dots, n\}$ and vanishes otherwise. Note that $\hat{\delta}^{\alpha\beta}$ are just the coefficients of the matrix $\mathbf{\Pi}$. Given stochastic integrals X, Y with matrices \mathbf{X}, \mathbf{Y} respectively, the coefficients of the product XY then form the matrix $\mathbf{XY} + \mathbf{XY} + \mathbf{X\Pi Y}$.

We may write the generator of the stochastic evolution as $dG(t) = G_{\alpha\beta}dA^{\alpha\beta}(t)$ and the isometry and co-isometry conditions are

$$\mathbf{G} + \mathbf{G}^\dagger + \mathbf{G}^\dagger \mathbf{\Pi} \mathbf{G} = 0 = \mathbf{G} + \mathbf{G}^\dagger + \mathbf{G} \mathbf{\Pi} \mathbf{G}^\dagger. \quad (7)$$

The expression for \mathbf{G} above then gives the general solution to this equation. The Heisenberg equation for an initial operator $\mathbf{X} \in B(\mathfrak{h})$ is then

$$j_t(\mathbf{X}) = \mathbf{X} \otimes 1 + \int_0^t j_s(\mathcal{L}_{\alpha\beta}(\mathbf{X})) \otimes dA^{\alpha\beta}(s)$$

and $j_t(\mathcal{L}_{\alpha\beta}(X))$ are the components of a matrix $j_t(X\mathbf{G} + \mathbf{G}^\dagger X + \mathbf{G}^\dagger \Pi X \Pi \mathbf{G})$. The super-operators $\mathcal{L}_{\alpha\beta}$ are known as the Evans-Hudson maps.

The output processes are then $\tilde{A}^{\alpha\beta}(t) := V(t)^\dagger \Lambda^{\alpha\beta}(t) V(t)$ and we deduce that

$$d\tilde{A}^{\alpha\beta}(t) \equiv j_t(M_{\alpha\mu}^\dagger M_{\beta\nu}) dA^{\mu\nu}(t).$$

This invariance of time, $d\tilde{t} = dt$, is the motivation for the term ‘‘Galilean transformation’’, and we now explore some of its properties.

Definition 2 *The Galilean group $\mathfrak{Gal}(\mathfrak{h}, \mathfrak{K})$ is the group of operators of the form (6) in $B(\mathfrak{h} \otimes (\mathbb{C} \oplus \mathfrak{K}))$ where $\mathbf{S} \in B(\mathfrak{h} \otimes \mathfrak{K})$ unitary, $\mathbf{L} \in B(\mathfrak{h}, \mathfrak{h} \otimes \mathfrak{K})$.*

The group identity is \mathbf{I} and we readily observe the group laws

$$\begin{pmatrix} 1 & 0 \\ \mathbf{L}_1 & \mathbf{S}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathbf{L}_2 & \mathbf{S}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathbf{L}_1 + \mathbf{S}_1 \mathbf{L}_2 & \mathbf{S}_1 \mathbf{S}_2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ \mathbf{L} & \mathbf{S} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\mathbf{S}^\dagger \mathbf{L} & \mathbf{S}^\dagger \end{pmatrix}.$$

Proposition 3 *If $\mathbf{M} \in \mathfrak{Gal}(\mathfrak{h}, \mathfrak{K})$ then we have the identity $\mathbf{M} \mathbf{I} \mathbf{M}^\dagger = \Pi$.*

Proposition 4 *The set $\mathfrak{G}(\mathfrak{h}, \mathfrak{K})$ is invariant under the action $\mathbf{G} \mapsto \mathbf{N}^\dagger \mathbf{G} \mathbf{N}$ for all $\mathbf{N} \in \mathfrak{Gal}(\mathfrak{h}, \mathfrak{K})$.*

Proof. Let $\mathbf{G} \in \mathfrak{G}(\mathfrak{h}, \mathfrak{K})$ and $\mathbf{N} \in \mathfrak{Gal}(\mathfrak{h}, \mathfrak{K})$. Setting $\mathbf{G}' = \mathbf{N}^\dagger \mathbf{G} \mathbf{N}$ we see that

$$\begin{aligned} \mathbf{G}' + \mathbf{G}'^\dagger &= \mathbf{N}^\dagger (\mathbf{G} + \mathbf{G}^\dagger) \mathbf{N} = -\mathbf{N}^\dagger \mathbf{G} \Pi \mathbf{G}^\dagger \mathbf{N} \\ &\equiv -\mathbf{N}^\dagger \mathbf{G} \mathbf{N} \Pi \mathbf{N}^\dagger \mathbf{G}^\dagger \mathbf{N} = -\mathbf{G}' \Pi \mathbf{G}'^\dagger \end{aligned}$$

since $\mathbf{N} \Pi \mathbf{N}^\dagger = \Pi$. Similarly, $\mathbf{G}' + \mathbf{G}'^\dagger = -\mathbf{G}'^\dagger \Pi \mathbf{G}'$. ■

Proposition 5 *Let $\mathbf{G} \in \mathfrak{G}(\mathfrak{h}, \mathfrak{K})$ with associated Galilean transformation $\mathbf{M} = \mathbf{1} + \Pi \mathbf{G}$. Then*

$$\mathbf{M}^\dagger \mathbf{G} \mathbf{M} = \mathbf{G}.$$

Proof. This follows from the observation

$$(\mathbf{I} + \mathbf{G}^\dagger \Pi) \mathbf{G} (\mathbf{1} + \Pi \mathbf{G}) = \mathbf{G} + (\mathbf{G}^\dagger + \mathbf{G} + \mathbf{G}^\dagger \Pi \mathbf{G}) \Pi \mathbf{G}.$$

■

2.4 Hamiltonian Description of a Single Markov Component

Let us consider the strongly continuous one-parameter unitary group $U_0(t)$ performing the time shifts. For instance, taking $\varepsilon(f)$ to be the exponential vector with test function f , we have the action

$$U_0(t) u \otimes \varepsilon(f) = u \otimes \varepsilon(\vartheta_{-t}f)$$

where $\vartheta_t f(\cdot) \triangleq f(\cdot - t)$. The family $\{V(t) : t \geq 0\}$ forms a right unitary co-cycle with respect to U_0 , that is $V(t+s) = U_0(-s)V(t)U_0(s)V(s)$ ($t, s \geq 0$), and we obtain a strongly continuous unitary group U by setting

$$U(t) = \begin{cases} U_0(t)V(t), & t \geq 0, \\ V^\dagger(-t)U_0(t), & t < 0. \end{cases}$$

The generators of U and U_0 will be denoted as H and H_0 respectively. The problem of characterizing H for the class of Hudson and Parthasarathy quantum stochastic evolutions has been carried out only in relatively recent times [9][17] and we now recall its explicit construction.

A vector $\Phi \in \mathfrak{H}$ will be a sequence $(\Phi_m)_{m=0}^\infty$ where $\Phi_m = \Phi_m(t_1, \dots, t_m)$ is a $\mathfrak{h} \otimes \mathfrak{K}^m$ -valued function completely symmetric under interchange of its arguments and such that

$$\sum_m \frac{1}{m!} \int_{\mathbb{R}^m} \|\Phi_m(t_1, \dots, t_m)\|_{\mathfrak{h} \otimes \mathfrak{K}^m}^2 dt_1 \cdots dt_m < \infty.$$

Let $\mathbb{R}_* = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ and define the domain $W(\mathbb{R}_*, \mathfrak{K}, \mathfrak{h})$ consisting of vectors such that each Φ_m is differentiable in each of its arguments and that

- i) $\sum_{j=1}^m \frac{\partial}{\partial t_j} \Phi_m \in \mathfrak{h} \otimes L^2_{\mathfrak{K}}(\mathbb{R}_*^m)$, for each m ,
- ii) $\sum_m \frac{1}{m!} \int_{\mathbb{R}_*^m} \left\| \sum_{j=1}^m \frac{\partial}{\partial t_j} \Phi_m(t_1, \dots, t_m) \right\|_{\mathfrak{h} \otimes \mathfrak{K}^m}^2 dt_1 \cdots dt_m < \infty$,
- iii) $\lim_{t_{m+1} \rightarrow 0^\pm} \sum_m \frac{1}{m!} \int_{\mathbb{R}_*^m} \|\Phi_m(t_1, \dots, t_m, t_{m+1})\|_{\mathfrak{h} \otimes \mathfrak{K}^{m+1}}^2$ exists.

(Note that the left and right limits above need not coincide!)

The following operators $H_0, a(t)$ for $t \neq 0$, and $a(0^\pm)$, are introduced on the domain $W(\mathbb{R}_*, \mathfrak{K}, \mathfrak{h})$

$$\begin{aligned} (H_0 \Phi)_m &= i \sum_{j=1}^m \frac{\partial}{\partial t_j} \Phi_m, \\ (a_i(t) \Phi)_m(t_1, \dots, t_m) &= e_{i \lrcorner} \Phi_{m+1}(t_1, \dots, t_{m+1})|_{t=t_{m+1}}, \end{aligned}$$

where $e_{i \lrcorner}$ is the trace operation from $\mathfrak{h} \otimes \mathfrak{K}^{m+1}$ down to $\mathfrak{h} \otimes \mathfrak{K}^m$ with respect to $e_i \in \mathfrak{K}$, see [17] for more details. The operator H_0 on the Sobolev-Fock domain

$W(\mathbb{R}_*, \mathfrak{K}, \mathfrak{h})$ coincides with the generator H_0 of translation by time shift on the dense subset for which the right and left hand limits in iii) agree. More generally we have the relation, which is a consequence of integration by parts with a jump discontinuity at the origin, for $\Phi, \Psi \in W(\mathbb{R}_*, \mathfrak{K}, \mathfrak{h})$

$$\langle \Phi | H_0 \Psi \rangle = \langle H_0 \Phi | \Psi \rangle + i \sum_{j=1}^n \langle a_j(0^-) \Phi | a_j(0^-) \Psi \rangle - i \sum_{j=1}^n \langle a_j(0^+) \Phi | a_j(0^+) \Psi \rangle$$

with the sum over an arbitrary orthonormal basis $\{e_i\}$ for the multiplicity space \mathfrak{K} . Note that we may formally write [9]

$$H_0 \equiv \sum_{j=1}^n \int_{\mathbb{R}_*} a_j^\dagger(t) i \frac{\partial}{\partial t} a_j(t). \quad (8)$$

Next, fix a subset $D_{b.c.}(\mathbf{S}, \mathbf{L})$ of $W(\mathbb{R}_*, \mathfrak{K}, \mathfrak{h})$ consisting of those vectors Φ satisfying the boundary conditions

$$a_j(0^-) \Phi = S_{jk} a_k(0^+) \Phi + L_j \Phi. \quad (9)$$

The boundary condition can be written as a Galilean transformation

$$a_\alpha(0^-) \Phi = M_{\alpha\beta} a_\beta(0^+) \Phi$$

where we include the time case $a_0 = 1$.

Theorem 6 (Gregoratti) [17] *The Hamiltonian K associated with the quantum stochastic unitary process having the parameters $(\mathbf{S}, \mathbf{L}, \mathbf{H})$ has $\text{dom}(H) \cap W(\mathbb{R}_*, \mathfrak{K}, \mathfrak{h}) = D_{b.c.}(\mathbf{S}, \mathbf{L})$ and here it is given by*

$$H\Phi = \left(H_0 + \mathbf{H} - \frac{i}{2} L_j^\dagger L_j - i L_j^\dagger S_{jk} a_k(0^+) \right) \Phi.$$

The Hamiltonian H is essentially self-adjoint on this domain.

For our purposes, it is most convenient to write the equation for H and the boundary condition in terms of the model matrix \mathbf{V} as

$$\begin{aligned} -iH\Phi &= (V_{00} + V_{0k} a_k(0^+) - iH_0) \Phi \equiv (V_{0\beta} a_\beta(0^+) - iH_0) \Phi, \\ a_j(0^-) \Phi &= V_{j0} \Phi + V_{jk} a_k(0^+) \Phi \equiv V_{j\beta} a_\beta(0^+) \Phi. \end{aligned} \quad (10)$$

3 Quantum Networks

The situation sketched in figure 1 will be our simplest example of a quantum network: a single component system with one input channel terminating at an input port and one output channel starting at an output. A general quantum network will consist of several such components connected together and will typically have time delays, feedforward and feedback connections. The description will exhibit separate algebraic, topological and geometric content. While the abstract definition is rather involved, figure 2 below gives an example of the class of configurations that we wish to consider

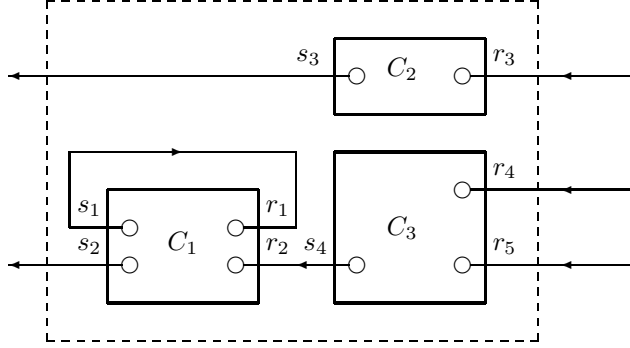


figure 2: A quantum network

We first list the basic features of a network. The network consists of a collection of components \mathcal{C} that are interconnected in a manner to be described shortly. Every component C will have at least one input port and one output port. Let us write $\mathcal{P}_{\text{in}}(C)$ and $\mathcal{P}_{\text{out}}(C)$ as the set of input and output ports, respectively, for component C . For each $r \in \mathcal{P}_{\text{in}}(C)$ we have an associated space $\mathfrak{K}_{\text{in}}^r$ which is the multiplicity space of the incoming channel. With a similar notation for the output ports, we impose the constraint

$$\mathfrak{K}_C \triangleq \bigoplus_{r \in \mathcal{P}_{\text{in}}(C)} \mathfrak{K}_{\text{in}}^r \equiv \bigoplus_{s \in \mathcal{P}_{\text{out}}(C)} \mathfrak{K}_{\text{out}}^s$$

which means that the total multiplicity space of all inputs into a component equals the corresponding output one. The sets of all input and output ports in the network are then $\mathcal{P}_{\text{in}} = \cup_C \mathcal{P}_{\text{in}}(C)$ and $\mathcal{P}_{\text{out}} = \cup_C \mathcal{P}_{\text{out}}(C)$ and we also have

$$\mathfrak{K}_{\text{net}} \triangleq \bigoplus_{r \in \mathcal{P}_{\text{in}}} \mathfrak{K}_{\text{in}}^r \equiv \bigoplus_{s \in \mathcal{P}_{\text{out}}} \mathfrak{K}_{\text{out}}^s. \quad (11)$$

Let $n_{\text{in}}(C)$ and $n_{\text{out}}(C)$ be the number of input and output ports respectively in a component and set $n_{\text{in}} = \sum_C n_{\text{in}}(C)$, $n_{\text{out}} = \sum_C n_{\text{out}}(C)$. Note that $n_{\text{in}}(C)$ and $n_{\text{out}}(C)$ are both non-zero though they need not coincide - though the total input/output multiplicities ($\dim \mathfrak{K}_C$) must be equal! External fields propagate into the network along input channels terminating at some of the input ports. Likewise, the output fields propagate along output channels starting at some of the output ports. Internally, we also have pairs of input and output ports connected by further channels. In this way every port is connected to exactly one channel. The set of interconnections is described by fixing subsets $\mathcal{R}_{\text{in}} \subset \mathcal{P}_{\text{in}}$ and $\mathcal{R}_{\text{out}} \subset \mathcal{P}_{\text{out}}$ of equal size and a bijection $\sigma : \mathcal{R}_{\text{out}} \mapsto \mathcal{R}_{\text{in}}$. The pair $(s, \sigma(s))$ then determines an internal channel from output port s to input port $r = \sigma(s)$.

Topologically we think of channels as edges and will frequently refer to them as such, and distinguish the input, output and internal edges. Each internal edge can be written as a pair $e = (s, r)$ - s is the *source* and r is the *range* - and we will have a corresponding fixed multiplicity space \mathfrak{K}_e associated with the

channel, and this must agree with both $\mathfrak{K}_{\text{out}}^s$ and $\mathfrak{K}_{\text{in}}^r$ providing an additional constraint. The remaining ports are $\mathcal{Q}_{\text{in}} = \mathcal{P}_{\text{in}} \setminus \mathcal{R}_{\text{in}}$ and $\mathcal{Q}_{\text{out}} = \mathcal{P}_{\text{out}} \setminus \mathcal{R}_{\text{out}}$. For each $r \in \mathcal{Q}_{\text{in}}$ we have a semi-infinite input edge terminating at that input port, and similar for each $s \in \mathcal{Q}_{\text{out}}$. We note the identity

$$\mathfrak{K}_{\text{ext}} \triangleq \bigoplus_{r \in \mathcal{Q}_{\text{in}}} \mathfrak{K}_{\text{in}}^r \equiv \bigoplus_{s \in \mathcal{Q}_{\text{out}}} \mathfrak{K}_{\text{out}}^s$$

which implies that the total multiplicity of all external inputs equals that of all the outputs. The complete set of edges, including both internal and external, will be denoted as \mathcal{E} .

(In the network sketched in figure 2, we have three components $\mathcal{C} = \{C_1, C_2, C_3\}$ with $\mathcal{P}_{\text{in}}(C_1) = \{r_1, r_2\}$, $\mathcal{P}_{\text{out}}(C_1) = \{s_1, s_2\}$, $\mathcal{P}_{\text{in}}(C_2) = \{r_3\}$, $\mathcal{P}_{\text{out}}(C_2) = \{s_3\}$ and $\mathcal{P}_{\text{in}}(C_3) = \{r_4, r_5\}$, $\mathcal{P}_{\text{out}}(C_3) = \{s_4, s_5\}$. The total network has $\mathcal{P}_{\text{in}} = \{r_1, r_2, r_3, r_4, r_5\}$, $\mathcal{P}_{\text{out}} = \{s_1, s_2, s_3, s_4\}$. The interconnections are via internal edges $e_1 = (s_1, r_1)$ and $e_2 = (s_4, r_2)$ and so $\mathcal{R}_{\text{in}} = \{r_1, r_2\}$, $\mathcal{R}_{\text{out}} = \{s_1, s_4\}$ with $\sigma(s_1) = r_1$ and $\sigma(s_4) = r_2$, and $\mathcal{Q}_{\text{in}} = \{r_3, r_4, r_5\}$, $\mathcal{Q}_{\text{out}} = \{s_2, s_3\}$.)

In addition to this topological description, we also provide the arc-time taken to travel along each internal channel: this determines the various time delays in the network. We geometrize the edges by applying a local arctime coordinate to each one. Let $e = (s, r)$ be an internal edge then we may assign an arctime parameter t_e with range (T_r, T_s) with $T_s - T_r$ being the time taken to travel from output port s to input port r . For input channels terminating at $r \in \mathcal{Q}_{\text{in}}$ we have an arctime parameter over the semi-infinite range (T_r, ∞) and likewise for output channels leaving $s \in \mathcal{Q}_{\text{out}}$ we have an arctime parameter over the semi-infinite range $(-\infty, T_s)$. The one-particle Hilbert space for the field quanta in an edge $e \in \mathcal{E}$ of the network is then

$$L^2_{\mathfrak{K}_e}(e) \triangleq \begin{cases} L^2_{\mathfrak{K}_{\text{in}}^r}(T_r, \infty), & e \text{ an incoming edge terminating at } r \in \mathcal{Q}_{\text{in}}, \\ L^2_{\mathfrak{K}_{\text{out}}^s}(-\infty, T_s), & e \text{ an outgoing edge starting at } s \in \mathcal{Q}_{\text{out}}, \\ L^2_{\mathfrak{K}_e}(T_{r(e)}, T_{s(e)}), & e = (s(e), r(e)) \text{ an internal edge.} \end{cases}$$

We set

$$L^2(\mathcal{E}) \triangleq \bigoplus_{e \in \mathcal{E}} L^2_{\mathfrak{K}_e}(e)$$

The Fock space over all these spaces will be denoted as $\mathfrak{F}_{\mathcal{E}}$ and by the functorial property factors as

$$\mathfrak{F}_{\mathcal{E}} \triangleq \Gamma(L^2(\mathcal{E})) = \bigotimes_{e \in \mathcal{E}} \mathfrak{F}_e.$$

The Hilbert space for the entire network \mathcal{N} will then take the form

$$\mathfrak{H}_{\mathcal{N}} \triangleq \mathfrak{h} \otimes \mathfrak{F}_{\mathcal{E}},$$

where \mathfrak{h} is the Hilbert space for all the quantum mechanical degrees of freedom of the network components.

In figure 2, we have only sketched the interactions between components that are mediated by the channels. Even though it may appear that the network is

disconnected, the components may still be coupled, say by a Hamiltonian interaction. We have also stopped short of requiring that the space of components factors as, say, $\mathfrak{h} = \otimes_C \mathfrak{h}_C$ and it is convenient not to impose this at this stage.

Up to this point, we have described the flow through the channels from output to input ports. It still remains to describe the trans-component flow. This involves the boundary conditions relating the inputs to the outputs at each component. The most convenient way to describe this is through the notion of the model matrix. With each component C we associate a model matrix $\mathbf{V}_C \in \mathfrak{M}(\mathfrak{h}, \mathfrak{K}_C)$.

Definition 7 *Let $\mathbf{V}_i \in \mathfrak{M}(\mathfrak{h}, \mathfrak{K}_i)$ for $i = 1, 2$, then the concatenation of model matrices is $\mathbf{V}_1 \boxplus \mathbf{V}_2 \in \mathfrak{M}(\mathfrak{h}, \mathfrak{K}_1 \oplus \mathfrak{K}_2)$ defined by*

$$\begin{aligned} & \begin{pmatrix} -\frac{1}{2}\mathbf{L}_1^\dagger \mathbf{L}_1 - i\mathbf{H}_1 & -\mathbf{L}_1^\dagger \mathbf{S}_1 \\ \mathbf{L}_1 & \mathbf{S}_1 \end{pmatrix} \boxplus \begin{pmatrix} -\frac{1}{2}\mathbf{L}_2^\dagger \mathbf{L}_2 - i\mathbf{H}_2 & -\mathbf{L}_2^\dagger \mathbf{S}_2 \\ \mathbf{L}_2 & \mathbf{S}_2 \end{pmatrix} \\ \triangleq & \begin{pmatrix} -\frac{1}{2}\mathbf{L}_1^\dagger \mathbf{L}_1 - \frac{1}{2}\mathbf{L}_2^\dagger \mathbf{L}_2 - i(\mathbf{H}_1 + \mathbf{H}_2) & -\mathbf{L}_1^\dagger \mathbf{S}_1 & -\mathbf{L}_2^\dagger \mathbf{S}_2 \\ \mathbf{L}_1 & \mathbf{S}_1 & 0 \\ \mathbf{L}_2 & 0 & \mathbf{S}_2 \end{pmatrix}. \end{aligned}$$

Given the set of component model matrices we may then define the network model matrix to be

$$\mathbf{V} = \boxplus_{C \in \mathcal{C}} \mathbf{V}_C \in \mathfrak{M}(\mathfrak{h}, \mathfrak{K}_{\text{net}}).$$

The network model matrix \mathbf{V} relates input ports to output ports and is introduced independently of the interconnections which go from output ports to input ports. It takes the standard form

$$\mathbf{V} = \begin{pmatrix} -\frac{1}{2}\mathbf{L}^\dagger \mathbf{L} - i\mathbf{H} & -\mathbf{L}^\dagger \mathbf{S} \\ \mathbf{L} & \mathbf{S} \end{pmatrix}$$

and, explicitly, for $r \in \mathcal{P}_{\text{in}}$ and $s \in \mathcal{P}_{\text{out}}$, we have components

$$\begin{aligned} \mathbf{V}_{sr} &\equiv \mathbf{S}_{sr} : \mathfrak{h} \otimes \mathfrak{K}_{\text{in}}^r \mapsto \mathfrak{h} \otimes \mathfrak{K}_{\text{out}}^s, \\ \mathbf{V}_{s0} &\equiv \mathbf{L}_s : \mathfrak{h} \mapsto \mathfrak{h} \otimes \mathfrak{K}_{\text{out}}^s, \\ \mathbf{V}_{0r} &\equiv - \sum_{s \in \mathcal{P}_{\text{out}}} \mathbf{L}_s^\dagger \mathbf{S}_{sr} : \mathfrak{h} \otimes \mathfrak{K}_{\text{in}}^r \mapsto \mathfrak{h}, \\ \mathbf{V}_{00} &\equiv -\frac{1}{2} \sum_{s \in \mathcal{P}_{\text{out}}} \mathbf{L}_s^\dagger \mathbf{L}_s - i\mathbf{H} : \mathfrak{h} \otimes \mathfrak{K}_{\text{in}}^r \mapsto \mathfrak{h}. \end{aligned}$$

It is convenient to adopt the following block matrix representation with respect to these decompositions (11) of $\mathfrak{K}_{\text{net}}$

$$\mathbf{S} \equiv \begin{pmatrix} \mathbf{S}_{11} & \cdots & \mathbf{S}_{1n_{\text{in}}} \\ \vdots & \ddots & \vdots \\ \mathbf{S}_{n_{\text{out}}1} & \cdots & \mathbf{S}_{n_{\text{out}}n_{\text{in}}} \end{pmatrix}, \quad \mathbf{L} \equiv \begin{pmatrix} \mathbf{L}_1 \\ \vdots \\ \mathbf{L}_{n_{\text{out}}} \end{pmatrix}.$$

We require that the operators S_{sr} are contractions with S unitary on $\mathfrak{h} \otimes \mathfrak{K}_{\text{total}}$, and so

$$\sum_{s \in \mathcal{P}_{\text{out}}} S_{sr}^\dagger S_{sr'} = \delta_{rr'}, \quad \sum_{r \in \mathcal{P}_{\text{in}}} S_{sr} S_{s'r}^\dagger = \delta_{ss'}.$$

It is possible to use the definition of concatenation in reverse in order to analyze a network model matrix into irreducible components. This process is substantially more complicated as we have to consider all decompositions (11).

A vector $\Phi \in \mathfrak{H}_{\mathcal{N}}$ can be represented as follows: for each integer $m \geq 0$ we choose m locations on the edges \mathcal{E} with local arctime coordinates $\{t_1, \dots, t_m\}$, say with t_j on edge $e(j)$, to get a vector $\Phi_m(t_1, \dots, t_m) \in \mathfrak{h} \otimes (\otimes_{j=1}^m \mathfrak{K}_{e(j)})$ which we view as a subset of $\mathfrak{h} \otimes L^2(\mathcal{E}^m)$, where \mathcal{E}^m denotes the m -fold Cartesian product of \mathcal{E} . We now generalize the class of Sobolev-Fock vectors to networks.

Definition 8 *The class $W(\mathcal{N})$ of Sobolev-Fock functions over a network \mathcal{N} is the set of vectors $(\Phi_m)_{m=0}^\infty$ in $\mathfrak{H}_{\mathcal{N}}$ such that each Φ_m is differentiable in each of its arguments and that*

- i) $\sum_{j=1}^m \frac{\partial}{\partial t_j} \Phi_m \in \mathfrak{h} \otimes L^2(\mathcal{E}^m)$, for each m ,
- ii) $\sum_m \frac{1}{m!} \left\| \sum_{k=1}^m \frac{\partial}{\partial t_k} \Phi_m \right\|_{\mathfrak{h} \otimes L^2(\mathcal{E}^m)}^2 < \infty$,
- iii) $\lim_{t_{m+1} \rightarrow 0^\pm} \sum_m \frac{1}{m!} \left\| \Phi_{m+1}(\cdot, \dots, \cdot, t_{m+1}) \right\|_{\mathfrak{h} \otimes L^2(\mathcal{E}^{m+1})}^2$ exists.

For each internal edge $e = (r, s)$ we introduce the local annihilator density $a_{e,j}(t)$ on the domain of Sobolev-Fock associated with the edge and the j th vector of an orthonormal basis $\{e_j\}$ for the multiplicity space \mathfrak{K}_e and $T_r < t < T_s$. We also consider the one-sided limits T_r^+ and T_s^- and collect into the column-block vector of annihilators

$$a_e(t) \triangleq \begin{pmatrix} a_{e,1}(t) \\ \vdots \\ a_{e,n}(t) \end{pmatrix}, \quad a_r(T_r^+) \triangleq a_e(T_r^+), \quad a_s(T_s^-) \triangleq a_e(T_s^-),$$

where $n = \dim \mathfrak{K}_e$. We then define the operator for each internal edge $e = (r, s)$

$$H_0(e) = \sum_j \int_{T_r}^{T_s} a_{e,j}^\dagger(t) i \frac{\partial}{\partial t} a_{e,j}(t) dt = \int_{T_r}^{T_s} a_e^\dagger(t) i \frac{\partial}{\partial t} a_e(t) dt.$$

For the semi-finite external edges we have the corresponding expression with either upper or lower limit extended to infinity as appropriate. The total contribution is then $H_0 = \sum_{e \in \mathcal{E}} H_0(e)$ and this generalizes the operator (8). We will denote the ampliation of these operators up to $\mathfrak{h} \otimes \mathfrak{F}$ by the same symbol, and recall our earlier convention that $a_0 \equiv 1$.

Definition 9 The Hamiltonian $H_{\mathcal{N}}$ for a quantum network \mathcal{N} with model matrix \mathbf{V} is the operator on $\mathfrak{H}_{\mathcal{N}}$ given by the specification that all Sobolev-Fock vectors Φ in its domain satisfy the system of boundary conditions

$$a_s(T_s^-)\Phi = \sum_{r \in \{0\} \cup \mathcal{P}_{in}} V_{sr} a_r(T_r^+)\Phi,$$

for each component $s \in \mathcal{P}_{out}$, and on such vectors we have the action

$$-iH_{\mathcal{N}}\Phi = \sum_{r \in \{0\} \cup \mathcal{P}_{in}} V_{0r} a_r(T_r^+)\Phi - iH_0\Phi.$$

Lemma 10 The operator $H_{\mathcal{N}}$ in the definition of a quantum network Hamiltonian is symmetric on the restricted domain $\text{dom}(H_{\mathcal{N}}) \cap W(\mathcal{N})$.

Proof. We now have the network integration-by-parts formula

$$\langle \Phi | H_0 \Psi \rangle - \langle H_0 \Phi | \Psi \rangle = i \sum_{s \in \mathcal{P}_{out}} \langle a_s(T_s^-)\Phi | a_s(T_s^-)\Psi \rangle - i \sum_{r \in \mathcal{P}_{in}} \langle a_r(T_r^+)\Phi | a_r(T_r^+)\Psi \rangle$$

for Sobolev-Fock vectors Φ, Ψ . $H_{\mathcal{N}}$ is symmetric on the set of Sobolev-Fock vectors satisfying the boundary conditions, indeed,

$$\begin{aligned} \langle \Phi | H_{\mathcal{N}} \Psi \rangle - \langle H_{\mathcal{N}} \Phi | \Psi \rangle &= i \langle \Phi | V_{00} \Psi \rangle - i \langle V_{00} \Phi | \Psi \rangle + i \sum_{s \in \mathcal{P}_{out}} \langle a_s(T_s^-)\Phi | a_s(T_s^-)\Psi \rangle \\ &\quad + i \sum_{r \in \mathcal{P}_{in}} \{ \langle \Phi | V_{0r} a_r(T_r^+)\Psi \rangle - \langle V_{0r} a_r(T_r^+)\Phi | \Psi \rangle - \langle a_r(T_r^+)\Phi | a_r(T_r^+)\Psi \rangle \} \end{aligned}$$

and, substituting in for $a_s(T_s^-)\Psi$ and $a_s(T_s^-)\Phi$ and using the identities $V_{00} + V_{00}^\dagger + \sum_{s \in \mathcal{P}_{out}} V_{s0}^\dagger V_{s0} = 0$, $V_{0r} + \sum_{s \in \mathcal{P}_{out}} V_{s0}^\dagger V_{sr}$, we find the right hand side vanishes. ■

We may rewrite these equations alternatively as

$$\begin{aligned} -iH\Phi &= -\left(\frac{1}{2} \sum_{s \in \mathcal{P}_{out}} L_s^\dagger L_s + iH + iH_0\right)\Phi - \sum_{s \in \mathcal{P}_{out}} \sum_{r \in \mathcal{P}_{in}} L_s^\dagger S_{sr} a_r(T_r^+)\Phi. \\ a_s(T_s^-)\Phi &= \sum_{r \in \mathcal{P}_{in}} S_{sr} a_r(T_r^+)\Phi + L_s\Phi. \end{aligned}$$

We wish to identify $H_{\mathcal{N}}$ as the physical Hamiltonian for the network, that is, show that it is essentially self-adjoint. This is a reasonable assumption given that it is a direct second quantization of quantum graph models encountered in the literature. At the moment, we do not have a proof of this fact but formulate it as a conjecture.

Conjecture 11 We now assume that $H_{\mathcal{N}}$ defines an essentially self-adjoint operator.

The Hamiltonian $H_{\mathcal{N}}$ contains all the physical information about the network, including the interactions and interconnections. Generally speaking, the wave operator $V = U_0^\dagger U$ will not determine a quantum stochastic evolution of Hudson Parthasarathy type as the model is typically no longer Markovian. We shall show that, in the limit in which time delays along the internal paths vanish, we recover a Markovian model which is easily identifiable from H and which will have the Chebotarev-Gregoratti form.

4 Eliminating Internal Edges in the Zero Time Delay Limit

In this section we show how simpler Markovian models can be obtained by eliminating edges in a zero time delay limit. We achieve this by showing first how to eliminate one edge, and then showing that all edges may be eliminated, with the final Markovian model independent of the order in which the eliminations were performed.

Theorem 12 *Let $e_0 = (r_0, s_0)$ be an internal channel with time delay $\tau_0 = T_{s_0} - T_{r_0} \geq 0$ in a quantum network \mathcal{N} for which $1 - V_{s_0 r_0}$ is invertible. In the limit $\tau_0 \rightarrow 0^+$, the network reduces to \mathcal{N}_{red} in which the input and output ports are $\mathcal{P}_{in} \setminus \{r_0\}$ and $\mathcal{P}_{out} \setminus \{s_0\}$ and the edge e_0 eliminated. (In the case where r_0 and s_0 are initially in different components, then the components merge.) The reduced model matrix V^{red} then has the components*

$$V_{\alpha\beta}^{red} = V_{\alpha\beta} + V_{\alpha r_0} (1 - V_{s_0 r_0})^{-1} V_{s_0 \beta}, \quad (12)$$

for $\beta \in \{0\} \cup \mathcal{P}_{in} \setminus \{r_0\}$ and $\alpha \in \{0\} \cup \mathcal{P}_{out} \setminus \{s_0\}$.

Proof. For Φ in the class of Sobolev-Fock vectors we have that $a_{s_0} (T_{s_0}^-) \Phi$ will be norm convergent to $a_{r_0} (T_{r_0}^+) \Phi$ as $\tau_0 \rightarrow 0^+$. In this limit we therefore identify the values $a_{r_0} (T_{r_0}^+) \Phi$ and $a_{s_0} (T_{s_0}^-) \Phi$ at the start and end of the edge being eliminated. The boundary condition $a_{s_0} (T_{s_0}^-) \Phi = \sum_{\beta \in \{0\} \cup \mathcal{P}_{in}} V_{s_0 r} a_r (T_r^+) \Phi$ may then be rewritten as

$$a_{r_0} (T_{r_0}^+) \Phi = (1 - V_{s_0 r_0})^{-1} \sum_{r \in \{0\} \cup \mathcal{P}_{in} \setminus \{r_0\}} V_{s_0 r} a_r (T_r^+) \Phi$$

and substituting into the boundary condition for $s \in \mathcal{P}_{out} \setminus \{s_0\}$ yields

$$\begin{aligned} a_s (T_s^-) \Phi &= \sum_{r \in \{0\} \cup \mathcal{P}_{in}} V_{sr} a_r (T_r^+) \Phi \\ &= V_{s_0 s} \Phi + \sum_{r \in \mathcal{P}_{in} \setminus \{r_0\}} V_{s_0 r} a_r (T_r^+) \Phi + V_{s_0 r_0} a_{r_0} (T_{r_0}^+) \Phi \\ &\equiv \sum_{r \in \{0\} \cup \mathcal{P}_{in} \setminus \{r_0\}} V_{s_0 r}^{red} a_r (T_r^+) \Phi \end{aligned}$$

The reduced Hamiltonian is then defined by

$$\begin{aligned}
-iH^{\text{red}}\Phi &= \sum_{r \in \{0\} \cup \mathcal{P}_{\text{in}}} V_{0r} a_r (T_r^+) \Phi - iH_0^{\text{red}}\Phi \\
&= \sum_{r \in \{0\} \cup \mathcal{P}_{\text{in}} \setminus \{r_0\}} V_{0r} a_r (T_r^+) \Phi + V_{0r_0} a_{r_0} (T_{r_0}^+) \Phi - iH_0^{\text{red}}\Phi \\
&\equiv \sum_{r \in \{0\} \cup \mathcal{P}_{\text{in}} \setminus \{r_0\}} V_{0r}^{\text{red}} a_r (T_r^+) \Phi - iH_0^{\text{red}}\Phi,
\end{aligned}$$

where $H_0^{\text{red}} = \sum_{e \in \mathcal{E} \setminus \{e_0\}} H_0(e)$.

The above argument rests on the fact that $\|a_{r_0}(T_{r_0}^+)\Phi - a_{s_0}(T_{s_0}^-)\Phi\| \rightarrow 0$ as $\tau_0 \rightarrow 0$ for Sobolev-Fock vectors Φ . More explicitly, by rescaling the local time-coordinate on the edge to have fixed range $[0, 1]$, we encounter a sequence of Hamiltonians $H = H^{(\tau_0)}$ with common domain in $\mathfrak{H}_{\mathcal{N}}$. Let $U^{(\tau_0)}$ and U^{red} be the one-parameter unitary groups generated by the $H^{(\tau_0)}$ and H^{red} respectively. The Hamiltonian H^{red} has domain in $\mathfrak{H}_{\mathcal{N}^{\text{red}}} \subset \mathfrak{H}_{\mathcal{N}}$ with the embedding implemented in the obvious way by taking zero quanta in the eliminated edge. For each $\Phi \in \text{dom}(H^{\text{red}})$ we may construct a sequence $\Phi^{(\tau_0)} \in \text{dom}(H^{(\tau_0)})$ strongly convergent to Φ such that $H^{(\tau_0)}\Phi^{(\tau_0)}$ converges strongly to $H^{\text{red}}\Phi$. By the Trotter-Kato theorem, e.g. [10] theorem 3.17, we then have that

$$\lim_{\tau_0 \rightarrow 0} \sup_{0 \leq t \leq T} \left\| \left(U_t^{(\tau_0)} - U_t^{\text{red}} \right) \Phi \right\| = 0,$$

for all $T > 0$ and $\Phi \in \mathfrak{H}_{\mathcal{N}^{\text{red}}}$. ■

The reduced model matrix (12) is closely related to fractional linear transformations, see for instance [24], [16], [26, Chapter 10], and we introduce appropriate definitions in the present context.

Definition 13 Let \mathfrak{K}_e be the multiplicity space of an edge $e = (s, r)$ so that \mathfrak{K}_e is a subspace of the network multiplicity $\mathfrak{K}_{\text{net}}$ and let $X \in B(\mathfrak{K}_e)$. The feedback reduction of $\mathbf{V} \in \mathfrak{M}(\mathfrak{h}, \mathfrak{K})$ through the edge e , with gain X , is the map

$$\begin{aligned}
\mathcal{F}_e &: \mathfrak{M}(\mathfrak{h}, \mathfrak{K}) \times B(\mathfrak{K}_e) \mapsto \mathfrak{M}(\mathfrak{h}, \mathfrak{K} \ominus \mathfrak{K}_e) \\
&: (\mathbf{V}, X) \mapsto \mathcal{F}_e(\mathbf{V}, X)
\end{aligned}$$

where in terms of block decomposition we have

$$\mathcal{F}_e(\mathbf{V}, X)_{\alpha\beta} \triangleq V_{\alpha\beta} + V_{\alpha r} X (1 - V_{sr} X)^{-1} V_{s\beta}, \quad (13)$$

where the indices are $\beta \in \{0\} \cup P_{\text{in}} \setminus \{r_0\}$ and $\alpha \in \{0\} \cup P_{\text{out}} \setminus \{s_0\}$. The domain is then the set of all pairs (\mathbf{V}, X) such that $1 - V_{sr} X$ is invertible in $B(\mathfrak{K}_e)$. In the special case of unit gain we write $\mathcal{F}_e(T) \triangleq \mathcal{F}_e(T, 1)$. For fixed \mathbf{V} , the map $\mathcal{F}_e(\mathbf{V}, \cdot)$ is a non-commutative fractional linear, or Möbius, transformation.

Lemma 14 (Siegel) *If $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a unitary operator on the direct sum $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ of two Hilbert spaces with $\|A\| < 1$. Let $\Phi_T(X) \triangleq D + CX(1 - AX)^{-1}B$ with $X \in \text{Dom}(\Phi_S)$ whenever $\|X\| \leq 1$. For $X, Y \in \text{Dom}(\Phi_S)$ we have the Siegel identities.*

$$\begin{aligned}\Phi_S(X)^\dagger \Phi_S(Y) - 1 &= B^\dagger (1 - X^\dagger A^\dagger)^{-1} (X^\dagger Y - 1) (1 - AY) B, \\ \Phi_S(X) \Phi_S(Y)^\dagger - 1 &= C^\dagger (1 - XA)^{-1} (XY^\dagger - 1) (1 - A^\dagger Y) C^\dagger.\end{aligned}$$

A proof can be found in [24].

Corollary 15 *If $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is unitary and $1 - A$ is invertible, then $D + C(1 - A)^{-1}B$ is unitary.*

For consistency, we would hope that the reduced model matrix belongs to the class of model matrices with multiplicity space one dimension lower. This we now show to be the case.

Lemma 16 *Let \mathbf{V} be the model matrix determined by the operators (S, L, H) . Then the reduced model matrix \mathbf{V}^{red} obtained by eliminating the edge (r_0, s_0) is determined by the operators $(S^{\text{red}}, L^{\text{red}}, H^{\text{red}})$ where*

$$\begin{aligned}S_{sr}^{\text{red}} &= S_{sr} + S_{sr_0} (1 - S_{s_0 r_0})^{-1} S_{s_0 r}, \\ L_s^{\text{red}} &= L_s + S_{sr_0} (1 - S_{s_0 r_0})^{-1} L_{s_0}, \\ H^{\text{red}} &= H + \sum_{s \in \mathcal{P}_{\text{out}}} \text{Im } L_s^\dagger S_{sr_0} (1 - S_{s_0 r_0})^{-1} L_{s_0},\end{aligned}$$

for $r \in \mathcal{P}_{\text{in}} \setminus \{r_0\}$ and $s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}$.

Proof. The identifications $V_{sr}^{\text{red}} = S_{sr}^{\text{red}} = S_{sr} + S_{sr_0} (1 - S_{s_0 r_0})^{-1} S_{s_0 r}$ and $V_{s_0}^{\text{red}} = L_s^{\text{red}} = L_s + S_{sr_0} (1 - S_{s_0 r_0})^{-1} L_{s_0}$ are immediate. Unitarity of S^{red} follows from the above corollary to the Siegel identities. We next check that $V_{0r}^{\text{red}} \equiv - \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} L_s^{\text{red} \dagger} S_{sr}^{\text{red}}$. Here

$$\sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} L_s^{\text{red} \dagger} S_{sr}^{\text{red}} = \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} L_s^\dagger S_{sr}^{\text{red}} + \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} L_{s_0}^\dagger (1 - S_{s_0 r_0}^\dagger)^{-1} S_{s_0 r}^{\text{red}}$$

and we use the simplification

$$\begin{aligned}\sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} S_{s_0 r}^\dagger S_{sr}^{\text{red}} &= \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \left(S_{s_0 r}^\dagger S_{sr} + S_{s_0 r}^\dagger S_{sr_0} (1 - S_{s_0 r_0})^{-1} S_{s_0 r} \right) \\ &= \left(-S_{s_0 r_0}^\dagger S_{s_0 r} + (1 - S_{s_0 r_0}^\dagger S_{s_0 r_0}) (1 - S_{s_0 r_0})^{-1} S_{s_0 r} \right) \\ &= \left(-S_{s_0 r_0}^\dagger (1 - S_{s_0 r_0}) + (1 - S_{s_0 r_0}^\dagger S_{s_0 r_0}) \right) (1 - S_{s_0 r_0})^{-1} S_{s_0 r} \\ &= (1 - S_{s_0 r_0}^\dagger) (1 - S_{s_0 r_0})^{-1} S_{s_0 r}\end{aligned}$$

so that

$$\sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\text{red}\dagger} \mathbf{S}_{sr}^{\text{red}} = \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\dagger} \mathbf{S}_{sr}^{\text{red}} + \mathbf{L}_{s_0}^{\dagger} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{S}_{s_0 r}.$$

On the other hand

$$\begin{aligned} \mathbf{V}_{0r}^{\text{red}} &= \mathbf{V}_{0r} + \mathbf{V}_{0r_0} (1 - \mathbf{V}_{s_0 r_0})^{-1} \mathbf{V}_{s_0 r} \\ &= - \sum_{s \in \mathcal{P}_{\text{out}}} \mathbf{L}_s^{\dagger} \mathbf{S}_{sr} - \sum_{s \in \mathcal{P}_{\text{out}}} \mathbf{L}_s^{\dagger} \mathbf{S}_{sr_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{S}_{s_0 r} \\ &= - \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\dagger} \mathbf{S}_{sr}^{\text{red}} - \mathbf{L}_{s_0}^{\dagger} \mathbf{S}_{s_0 r} - \mathbf{L}_{s_0}^{\dagger} \mathbf{S}_{s_0 r_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{S}_{s_0 r} \\ &= - \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\dagger} \mathbf{S}_{sr}^{\text{red}} - \mathbf{L}_{s_0}^{\dagger} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{S}_{s_0 r} \equiv - \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\text{red}\dagger} \mathbf{S}_{sr}^{\text{red}}. \end{aligned}$$

Finally we must check that $\mathbf{V}_{00}^{\text{red}} = -\frac{1}{2} \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\text{red}\dagger} \mathbf{L}_s^{\text{red}} - i\mathbf{H}^{\text{red}}$. Let us use this as the definition of \mathbf{H}^{red} then we have

$$\begin{aligned} -i\mathbf{H}^{\text{red}} &= \mathbf{V}_{00} + \mathbf{V}_{0r_0} (1 - \mathbf{V}_{s_0 r_0})^{-1} \mathbf{V}_{s_0 0} + \frac{1}{2} \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\text{red}\dagger} \mathbf{L}_s^{\text{red}} \\ &= (-i\mathbf{H} - \frac{1}{2} \sum_{s \in \mathcal{P}_{\text{out}}} \mathbf{L}_s^{\dagger} \mathbf{L}_s) - \sum_{s \in \mathcal{P}_{\text{out}}} \mathbf{L}_s^{\dagger} \mathbf{S}_{sr_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{L}_{s_0} \\ &\quad + \frac{1}{2} \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\text{red}\dagger} \mathbf{L}_s^{\text{red}} \\ &= -i\mathbf{H} - \frac{1}{2} \mathbf{L}_{s_0}^{\dagger} \mathbf{L}_{s_0} - \frac{1}{2} \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\dagger} \mathbf{S}_{sr_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{L}_{s_0} \\ &\quad + \frac{1}{2} \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_{s_0}^{\dagger} (1 - \mathbf{S}_{s_0 r_0}^{\dagger})^{-1} \mathbf{S}_{sr_0} \mathbf{L}_s - \mathbf{L}_{s_0}^{\dagger} \mathbf{S}_{s_0 r_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{L}_{s_0} \\ &\quad + \frac{1}{2} \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_{s_0}^{\dagger} (1 - \mathbf{S}_{s_0 r_0}^{\dagger})^{-1} \mathbf{S}_{sr_0}^{\dagger} \mathbf{S}_{sr_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{L}_{s_0}. \end{aligned}$$

We collected together several terms to get $-\frac{1}{2} \mathbf{L}_{s_0}^{\dagger} \mathbf{X} \mathbf{L}_{s_0}$ where

$$\begin{aligned} \mathbf{X} &= 1 + 2\mathbf{S}_{s_0 r_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} - \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} (1 - \mathbf{S}_{s_0 r_0}^{\dagger})^{-1} \mathbf{S}_{sr_0}^{\dagger} \mathbf{S}_{sr_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} \\ &= 1 + 2\mathbf{S}_{s_0 r_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} - (1 - \mathbf{S}_{s_0 r_0}^{\dagger})^{-1} (1 - \mathbf{S}_{s_0 r_0}^{\dagger} \mathbf{S}_{s_0 r_0}) (1 - \mathbf{S}_{s_0 r_0})^{-1} \\ &= (1 - \mathbf{S}_{s_0 r_0}^{\dagger})^{-1} (\mathbf{S}_{s_0 r_0} - \mathbf{S}_{s_0 r_0}^{\dagger}) (1 - \mathbf{S}_{s_0 r_0})^{-1}. \end{aligned}$$

It follows that

$$\mathbf{H}^{\text{red}} = \mathbf{H} + \text{Im} \sum_{s \in \mathcal{P}_{\text{out}} \setminus \{s_0\}} \mathbf{L}_s^{\dagger} \mathbf{S}_{sr_0} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{L}_{s_0} + \text{Im} \mathbf{L}_{s_0}^{\dagger} (1 - \mathbf{S}_{s_0 r_0})^{-1} \mathbf{L}_{s_0}$$

Note that $\text{Im } L_{s_0}^\dagger (1 - S_{s_0 r_0})^{-1} L_{s_0} = \text{Im } L_{s_0}^\dagger S_{s_0 r_0} (1 - S_{s_0 r_0})^{-1} L_{s_0}$ we obtain H^{red} as the self-adjoint operator given in the statement of the lemma. ■

Lemma 17 *Let $e_1 = (s_1, r_1)$ and $e_2 = (s_2, r_2)$ be a pair of edges in a network then*

$$\mathcal{F}_{e_1} \circ \mathcal{F}_{e_2} = \mathcal{F}_{e_2} \circ \mathcal{F}_{e_1} = \mathcal{F}_{e_1 \oplus e_2}. \quad (14)$$

Proof. Without loss of generality suppose that $e_1 = (1, 1)$ and $e_2 = (2, 2)$, then for $\alpha, \beta \notin \{1, 2\}$

$$(\mathcal{F}_{e_2} \circ \mathcal{F}_{e_1} \mathbf{V})_{\alpha\beta} = \hat{V}_{\alpha\beta} + \hat{V}_{\alpha 2} (1 - \hat{V}_{22})^{-1} \hat{V}_{2\beta}$$

where $\hat{V}_{\alpha\beta} = V_{\alpha\beta} + V_{\alpha 1} (1 - V_{11})^{-1} V_{1\beta}$, and so we have

$$(\mathcal{F}_{e_2} \circ \mathcal{F}_{e_1} \mathbf{V})_{\alpha\beta} = \hat{V}_{\alpha\beta} + \begin{pmatrix} V_{\alpha 1} & V_{\alpha 2} \end{pmatrix} Z \begin{pmatrix} V_{\alpha 1} \\ V_{\alpha 2} \end{pmatrix}$$

with

$$\begin{aligned} Z &= \begin{pmatrix} \frac{1}{1-V_{11}} + \frac{1}{1-V_{11}} V_{12} \frac{1}{1-\hat{V}_{22}} V_{21} \frac{1}{1-V_{11}} & \frac{1}{1-V_{11}} V_{12} \frac{1}{1-\hat{V}_{22}} \\ \frac{1}{1-\hat{V}_{22}} V_{21} \frac{1}{1-V_{11}} & \frac{1}{1-\hat{V}_{22}} \end{pmatrix} \\ &\equiv \begin{pmatrix} 1 - V_{11} & -V_{12} \\ -V_{21} & 1 - V_{22} \end{pmatrix}^{-1}. \end{aligned}$$

The expression is clearly symmetric under interchange of 1 and 2, and corresponds to the double edge elimination. ■

This implies that the order in which we apply zero time delay limits to eliminate multiple internal channels does not in fact matter, and can be combined simultaneously. In this manner, every quantum network may be reduced to a single Markovian component in a unique well-defined algebraically manner by eliminating the time delays in all internal channels by means of the map

$$\mathcal{F} = \circ_{e \in \mathcal{E}_{\text{int}}} \mathcal{F}_e : \mathfrak{M}(\mathfrak{h}, \mathfrak{K}_{\text{net}}) \mapsto \mathfrak{M}(\mathfrak{h}, \mathfrak{K}_{\text{ext}}).$$

If we decide to eliminate all internal channels by making the connections, inserting a gain matrix X , and taking the zero time-delay limit, then the resulting network will have model matrix $\mathcal{F}(\mathbf{V}, X)$ and of course has dimensions determined by the remaining (external) channels. The functorial property is that we are able to eliminate blocks of channels in one step, giving the same answer as if we performed the eliminations one-by-one.

5 Physical Applications

We shall be interested in the situation where we eliminate *all* the *internal channels* (total multiplicity space \mathfrak{K}_{i}) leaving only the external channels (total multiplicity space \mathfrak{K}_{e}). With respect to the decomposition $\mathfrak{K} = \mathfrak{K}_{\text{int}} \oplus \mathfrak{K}_{\text{ext}}$, we write

the operators (S, L, H) of the network as

$$S = \begin{pmatrix} S_{ii} & S_{ie} \\ S_{ei} & S_{ee} \end{pmatrix}, \quad L = \begin{pmatrix} L_i \\ L_e \end{pmatrix}.$$

The feedback reduced model matrix may be conveniently expressed as

$$\mathcal{F}(\mathbf{V}, \boldsymbol{\eta}^{-1})_{\alpha\beta} \triangleq V_{\alpha\beta} + V_{\alpha i} (\eta - V_{ii})^{-1} V_{i\beta}$$

for $\alpha, \beta \in \{0, \mathbf{e}\}$, where η is the (unitary) *adjacency matrix*

$$\eta_{sr} = \begin{cases} 1, & \text{if } (s, r) \text{ is an internal channel,} \\ 0, & \text{otherwise.} \end{cases}$$

Here the “gain” η is the set of instructions as to which internal output port gets connected up to which internal input port. We, of course, have $\eta = 1$ if we match up the labels of the input and output ports according to the connections, however, it is computationally easier to work with a general labelling and just specify the adjacency matrix. The reduced model matrix \mathbf{V}^{red} obtained by eliminating all the internal channels is determined by the operators $(S^{\text{red}}, L^{\text{red}}, H^{\text{red}})$ given by

$$\begin{aligned} S^{\text{red}} &= S_{ee} + S_{ei} (\eta - S_{ii})^{-1} S_{ie}, \\ L^{\text{red}} &= L_e + S_{ei} (\eta - S_{ii})^{-1} L_i, \\ H^{\text{red}} &= H + \sum_{i=\mathbf{i}, \mathbf{e}} \text{Im } L_j^\dagger S_{ji} (\eta - S_{ii})^{-1} L_i. \end{aligned}$$

5.1 Quantum Systems in Feedforward

Our first application is to derive the formula for systems in series. This is the simplest nontrivial example of a quantum network.

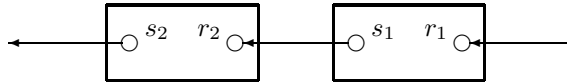


figure 3: Systems in series

We begin by taking (S_1, L_1, H_1) and (S_2, L_2, H_2) to be the operators of the first and second system when considered as separate systems driven by independent noises. The model matrix for the network is then

$$\mathbf{V} = \begin{pmatrix} -\sum_{j=1,2} (\frac{1}{2} L_j^\dagger L_j + i H_j) & -L_1^\dagger S_1 & -L_2^\dagger S_2 \\ L_1 & S_1 & 0 \\ L_2 & 0 & S_2 \end{pmatrix}$$

with respect to the labelling $s = \{0, s_1, s_2\}$ for the outputs (rows) and $r = \{0, r_1, r_2\}$ for the inputs (columns). We wish to reform a feedback reduction wherein we connect the systems via edge $e = (s_1, r_2)$ and take the zero time-delay limit along the edge. The resulting model should then be Markovian and its model matrix is given by

$$\begin{aligned} \mathbf{V}_{\text{series}} &= \mathcal{F}_e \mathbf{V} \\ &= \begin{pmatrix} -\sum_{j=1,2} (\frac{1}{2} \mathbf{L}_j^\dagger \mathbf{L}_j + i \mathbf{H}_j) & -\mathbf{L}_1^\dagger \mathbf{S}_1 \\ \mathbf{L}_2 & 0 \end{pmatrix} + \begin{pmatrix} -\mathbf{L}_2^\dagger \mathbf{S}_2 \\ \mathbf{S}_2 \end{pmatrix} (1-0)^{-1} (\mathbf{L}_1, \mathbf{S}_1) \\ &= \begin{pmatrix} -\sum_{j=1,2} (\frac{1}{2} \mathbf{L}_j^\dagger \mathbf{L}_j + i \mathbf{H}_j) - \mathbf{L}_2^\dagger \mathbf{S}_2 \mathbf{L}_1 & -\mathbf{L}_1^\dagger \mathbf{S}_1 - \mathbf{L}_2^\dagger \mathbf{S}_2 \mathbf{S}_1 \\ \mathbf{L}_2 + \mathbf{S}_2 \mathbf{L}_1 & \mathbf{S}_2 \mathbf{S}_1 \end{pmatrix}, \end{aligned}$$

that is, the operators determining the reduced system are

$$\begin{aligned} \mathbf{S}_{\text{series}} &= \mathbf{S}_2 \mathbf{S}_1, \\ \mathbf{L}_{\text{series}} &= \mathbf{L}_2 + \mathbf{S}_2 \mathbf{L}_1, \\ \mathbf{H}_{\text{series}} &= \mathbf{H}_1 + \mathbf{H}_2 + \text{Im} \left\{ \mathbf{L}_2^\dagger \mathbf{S}_2 \mathbf{L}_1 \right\}. \end{aligned}$$

The Evans-Hudson maps associated with the feedforward system can be related to those of the individual systems via the identity

$$\mathcal{L}_{\alpha\beta}(\cdot) = \mathcal{L}_{\alpha\beta}^{(1)}(\cdot) + (\mathbf{M}_{\mu\alpha}^{(1)})^\dagger \mathcal{L}_{\mu\nu}^{(2)}(\cdot) \mathbf{M}_{\nu\beta}^{(1)}. \quad (15)$$

The case most familiar to the quantum optics community is two cavity systems in cascade. Here $\mathbf{S}_1 = \mathbf{S}_2 = 1$ and $L_i = \sqrt{\gamma_i} a_i$ ($i = 1, 2$) so we obtain $L = \sqrt{\gamma_1} a_1 + \sqrt{\gamma_2} a_2$, $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 + \frac{1}{2i} \sqrt{\gamma_1 \gamma_2} (a_2^\dagger a_1 - a_1^\dagger a_2)$. This agrees with the calculations of Gardiner [11] for cascaded oscillators. Gardiner's derivation [11] of the cascade rule is different from ours, but would extend to cover the gauge case due to (15).

5.1.1 The Series Product

The rule for determining the form of the model for systems in series has been previously given for the Itô generator matrices where it was called the *series product*. It is related to the general question of how to “add” stochastic derivations in order to obtain a stochastic derivation [1]. We recall its definition and establish its basic properties.

Definition 18 *Let \mathbf{G}_1 and \mathbf{G}_2 be the Itô generator matrices with the same multiplicity spaces, then the series product is defined to be*

$$\mathbf{G}_2 \triangleleft \mathbf{G}_1 \triangleq \mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_2 \Pi \mathbf{G}_1.$$

Lemma 19 *The series product of two Itô matrices $\mathbf{G} = \mathbf{G}_2 \triangleleft \mathbf{G}_1$ is again an Itô matrix and if the \mathbf{G}_i have parameters $(\mathbf{S}_i, \mathbf{L}_i, \mathbf{H})$ then \mathbf{G} has parameters $(\mathbf{S}_{\text{series}}, \mathbf{L}_{\text{series}}, \mathbf{H}_{\text{series}})$. If \mathbf{M}_i is the Galilean matrix associated with \mathbf{G}_i then the Galilean matrix associated with $\mathbf{G} = \mathbf{G}_2 \triangleleft \mathbf{G}_1$ is $\mathbf{M} = \mathbf{M}_2 \mathbf{M}_1$. The series product is not symmetric, but is associative.*

Proof. One readily checks that the series product of two Itô matrices satisfies the conditions to be an Itô generating matrix. The specific form of the parameters is found by inspection. The associated Galilean transformation is

$$\begin{aligned} \mathbf{M} &= \mathbf{1} + \mathbf{\Pi}(\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_2 \mathbf{\Pi} \mathbf{G}_1) \\ &= \mathbf{1} + \mathbf{\Pi} \mathbf{G}_1 + \mathbf{\Pi} \mathbf{G}_2 + \mathbf{\Pi} \mathbf{G}_2 \mathbf{\Pi} \mathbf{G}_1 \\ &= (\mathbf{1} + \mathbf{\Pi} \mathbf{G}_2)(\mathbf{1} + \mathbf{\Pi} \mathbf{G}_1) \\ &= \mathbf{M}_2 \mathbf{M}_1. \end{aligned}$$

To prove associativity, let us construct the $(1 + n + 1)$ -square matrix

$$\mathbb{V} = \begin{pmatrix} 1 & -\mathbf{L}^\dagger \mathbf{S} & -\frac{1}{2} \mathbf{L}^\dagger \mathbf{L} - i\mathbf{H} \\ 0 & \mathbf{S} & \mathbf{L} \\ 0 & 0 & 1 \end{pmatrix} \quad (16)$$

from the model parameters $(\mathbf{S}, \mathbf{L}, \mathbf{H})$. Then the series product corresponds to the ordinary matrix product $\mathbb{V}_{\text{series}} = \mathbb{V}_2 \mathbb{V}_1$ which is clearly associative. ■

Associativity means that we can extend the result immediately to several systems cascaded in series. The easiest way to calculate the model matrix for several components in series is then by the ordinary matrix product of the augmented matrices

$$\mathbb{V}_{\text{series}} = \mathbb{V}_n \cdots \mathbb{V}_2 \mathbb{V}_1.$$

In the lemma, the matrix \mathbb{V} is of the type introduced by Belavkin [5] to efficiently

capture the Itô correction as an ordinary product. Let $\zeta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ then

we obtain an involution $\mathbb{X}^\star = \zeta \mathbb{X}^\dagger \zeta$ on the space of $(1 + n + 1)$ -dimensional matrices. Then we are considering precisely the class of \star -unitary matrices of

the form $\mathbb{V} = \begin{pmatrix} 1 & B & A \\ 0 & D & C \\ 0 & 0 & 1 \end{pmatrix}$, that is $\mathbb{V}^\star \mathbb{V} = \mathbb{V} \mathbb{V}^\star = 1$. In particular, the

product of two \star -unitaries is again a \star -unitary.

Remark 20 *Finally let us make the important remark that nowhere did we assume that the entries of \mathbf{G}_1 and \mathbf{G}_2 had to commute. This means that the series product can describe not only forward from one component system to an independent system, but also feedback into itself as well. This is captured in the picture below.*

5.2 Beam Splitters

A simple beam splitter is a device performing physical superposition of two input fields. It is described by a fixed unitary operator $T = \begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix} \in U(2)$:

$$\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

This is a canonical transformation and the output fields satisfy the same canonical commutation relations as the inputs. The action of the beam splitter is depicted in the figure below. On the left we have a traditional view of the two inputs (A_1, A_2) being split into two output fields (\tilde{A}_1, \tilde{A}_2). On the right we have our view of the beam splitter as being a component with two input ports and two output ports: we have sketched some internal detail to emphasize how the scattering (superimposing) of inputs however we shall usually just draw this as a “black box” component in the following.

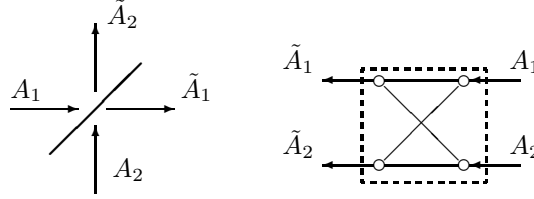


Figure 4: Beam-splitter component.

Our aim is to describe the effective Markov model for the feedback device sketched below where the feedback is achieved by means of a beam splitter. Here we have a component system, called the plant, in-loop and we assume that it is described by the parameters (S_0, L_0, H_0) . Markovianity here corresponds to the limit of instantaneous feedback.

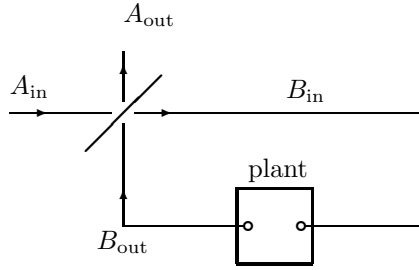


Figure 5: Feedback using a beam-splitter.

It is more convenient to view this as the network sketched below.

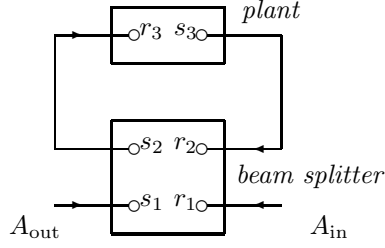


Figure 6: Network representation.

Here we have the pair of internal edges (s_2, r_3) and (s_3, r_2) . The model matrix for the network is

$$\mathbf{V} = \begin{pmatrix} -\frac{1}{2}\mathbf{L}_0^\dagger \mathbf{L}_0 - i\mathbf{H}_0 & 0 & 0 & -\mathbf{L}_0^\dagger \mathbf{S}_0 \\ 0 & T_{11} & T_{12} & 0 \\ 0 & T_{21} & T_{22} & 0 \\ \mathbf{L}_0 & 0 & 0 & \mathbf{S}_0 \end{pmatrix}$$

with respect to the labels $(0, s_1, s_2, s_3)$ for the rows and $(0, r_1, r_2, r_3)$ for the columns. Here we have

$$\begin{aligned} \mathbf{S}_{ii} &= \begin{pmatrix} T_{22} & 0 \\ 0 & \mathbf{S}_0 \end{pmatrix}, & \mathbf{S}_{ie} &= \begin{pmatrix} T_{21} \\ 0 \end{pmatrix}, \\ \mathbf{S}_{ei} &= (T_{12}, 0), & \mathbf{S}_{ee} &= T_{11}, \\ \mathbf{L}_i &= \begin{pmatrix} \mathbf{L}_0 \\ 0 \end{pmatrix}, & \mathbf{L}_e &= 0, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Note that the adjacency matrix has row indices (s_2, s_3) and columns indices (r_2, r_3) labelling the internal ports, and that the edges are the off diagonals (s_2, r_3) and (s_3, r_2) . We may perform the two eliminations simultaneously to obtain

$$\begin{aligned} \mathbf{S}_{\text{red}} &= T_{11} + (T_{12} \ 0) \begin{pmatrix} -T_{22} & 1 \\ 1 & -\mathbf{S}_0 \end{pmatrix}^{-1} \begin{pmatrix} T_{21} \\ 0 \end{pmatrix} \\ &= T_{11} + T_{12} (\mathbf{S}_0^{-1} - T_{22})^{-1} T_{21}, \\ \mathbf{L}_{\text{red}} &= (T_{12} \ 0) \begin{pmatrix} -T_{22} & 1 \\ 1 & -\mathbf{S}_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \mathbf{L}_0 \end{pmatrix} \\ &= T_{12} (1 - \mathbf{S}_0 T_{22})^{-1} \mathbf{L}_0, \\ \mathbf{H}_{\text{red}} &= \mathbf{H}_0 + \text{Im} \begin{pmatrix} 0 & \mathbf{L}_0^\dagger \end{pmatrix} \begin{pmatrix} -T_{22} & 1 \\ 1 & -\mathbf{S}_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \mathbf{L}_0 \end{pmatrix} \\ &= \mathbf{H}_0 + \text{Im} \mathbf{L}_0^\dagger (1 - \mathbf{S}_0 T_{22})^{-1} \mathbf{L}_0. \end{aligned}$$

5.3 The Redheffer Star Product

An important feedback arrangement is shown in the figure below.

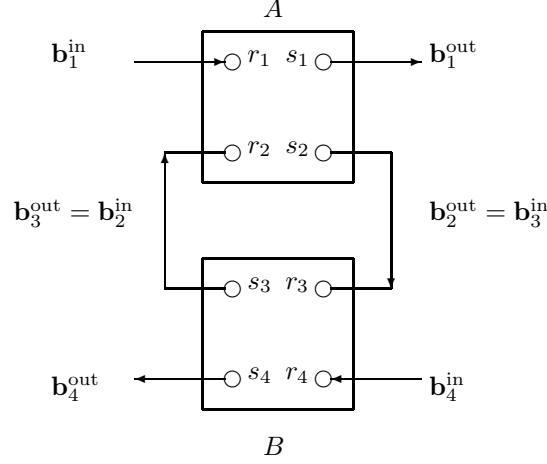


Figure 7 Composite System

We shall now derive this system taking component A to be described $\begin{pmatrix} S_{11}^A & S_{12}^A \\ S_{21}^A & S_{22}^A \end{pmatrix}$, $\begin{pmatrix} L_1^A \\ L_2^A \end{pmatrix}$, H_A and B by $\begin{pmatrix} S_{33}^B & S_{34}^B \\ S_{43}^B & S_{44}^B \end{pmatrix}$, $\begin{pmatrix} L_3^B \\ L_4^B \end{pmatrix}$, H_B . The operators of systems A are assumed to commute with those of B . We have two internal channels to eliminate which we can do in sequence, or simultaneously. We shall do the latter. here we have

$$\begin{aligned} S_{ee} &= \begin{pmatrix} S_{11}^A & 0 \\ 0 & S_{44}^B \end{pmatrix}, S_{ei} = \begin{pmatrix} S_{12}^A & 0 \\ 0 & S_{43}^B \end{pmatrix} \\ S_{ie} &= \begin{pmatrix} S_{21}^A & 0 \\ 0 & S_{34}^B \end{pmatrix}, S_{ii} = \begin{pmatrix} S_{22}^A & 0 \\ 0 & S_{33}^B \end{pmatrix} \end{aligned}$$

and

$$L_e = \begin{pmatrix} L_1^A \\ L_4^B \end{pmatrix}, \quad L_i = \begin{pmatrix} L_2^A \\ L_3^B \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The reduced operators are therefore

$$\begin{aligned} S_{\star} &= \begin{pmatrix} S_{11}^A & 0 \\ 0 & S_{44}^B \end{pmatrix} + \begin{pmatrix} S_{12}^A & 0 \\ 0 & S_{43}^B \end{pmatrix} \begin{pmatrix} -S_{22}^A & 1 \\ 1 & -S_{33}^B \end{pmatrix}^{-1} \begin{pmatrix} S_{21}^A & 0 \\ 0 & S_{34}^B \end{pmatrix} \\ &= \begin{pmatrix} S_{11}^A + S_{12}^A S_{33}^B (1 - S_{22}^A S_{33}^B)^{-1} S_{21}^A & S_{12}^A (1 - S_{22}^A S_{33}^B)^{-1} S_{34}^B \\ S_{43}^B (1 - S_{22}^A S_{33}^B)^{-1} S_{21}^A & S_{44}^B + S_{43}^B (1 - S_{22}^A S_{33}^B)^{-1} S_{22}^A S_{34}^B \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
\mathbf{L}_\star &= \begin{pmatrix} \mathbf{L}_1^A \\ \mathbf{L}_4^B \end{pmatrix} + \begin{pmatrix} \mathbf{S}_{12}^A & 0 \\ 0 & \mathbf{S}_{43}^B \end{pmatrix} \begin{pmatrix} -\mathbf{S}_{22}^A & 1 \\ 1 & -\mathbf{S}_{33}^B \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{L}_2^A \\ \mathbf{L}_3^B \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{L}_1^A + \mathbf{S}_{12}^A \mathbf{S}_{33}^B (1 - \mathbf{S}_{22}^A \mathbf{S}_{33}^B)^{-1} \mathbf{L}_2^A + \mathbf{S}_{12}^A (1 - \mathbf{S}_{22}^A \mathbf{S}_{33}^B)^{-1} \mathbf{L}_3^B \\ \mathbf{L}_4^B + \mathbf{S}_{43}^B (1 - \mathbf{S}_{22}^A \mathbf{S}_{33}^B)^{-1} \mathbf{L}_2^A + \mathbf{S}_{43}^B \mathbf{S}_{22}^A (1 - \mathbf{S}_{22}^A \mathbf{S}_{33}^B)^{-1} \mathbf{L}_3^B \end{pmatrix}, \\
\mathbf{H}_\star &= \mathbf{H}_A + \mathbf{H}_B + \text{Im} \left\{ \mathbf{L}_3^{B\dagger} (1 - \mathbf{S}_{33}^B \mathbf{S}_{22}^A)^{-1} \mathbf{L}_3^B + \mathbf{L}_3^{B\dagger} (1 - \mathbf{S}_{33}^B \mathbf{S}_{22}^A)^{-1} \mathbf{S}_{33}^B \mathbf{L}_2^A \right. \\
&\quad + \mathbf{L}_2^{A\dagger} (1 - \mathbf{S}_{22}^A \mathbf{S}_{33}^B)^{-1} \mathbf{S}_{22}^A \mathbf{L}_3^B + \mathbf{L}_2^{A\dagger} (1 - \mathbf{S}_{22}^A \mathbf{S}_{33}^B)^{-1} \mathbf{L}_2^A \\
&\quad + \mathbf{L}_1^{A\dagger} \mathbf{S}_{12}^A (1 - \mathbf{S}_{33}^B \mathbf{S}_{22}^A)^{-1} \mathbf{L}_3^B + \mathbf{L}_1^{A\dagger} \mathbf{S}_{12}^A (1 - \mathbf{S}_{33}^B \mathbf{S}_{22}^A)^{-1} \mathbf{S}_{33}^B \mathbf{L}_2^A \\
&\quad \left. + \mathbf{L}_4^{B\dagger} \mathbf{S}_{43}^B (1 - \mathbf{S}_{22}^A \mathbf{S}_{33}^B)^{-1} \mathbf{S}_{22}^A \mathbf{L}_3^B + \mathbf{L}_4^{B\dagger} \mathbf{S}_{43}^B (1 - \mathbf{S}_{22}^A \mathbf{S}_{33}^B)^{-1} \mathbf{L}_2^A \right\}.
\end{aligned}$$

6 Topological Rules for Quantum Networks

We wish to state algebraic rules for constructing the operators $\mathbf{S}_{\text{net}}, \mathbf{L}_{\text{net}}, \mathbf{H}_{\text{net}}$ covering all the examples of feedback networks studied so far. For a given network, we denote by \mathcal{P}_{out} the set of output ports and $\mathcal{P}_{\text{out}}^{\text{ext}}$ the subset of output ports having external output. With each $i \in \mathcal{P}_{\text{out}}$ there is an associated *port operator* L_i . Similarly we have \mathcal{P}_{in} and $\mathcal{P}_{\text{in}}^{\text{ext}}$ for the inputs.

For $\gamma = (i, j)$ an ordered pair of ports, we set

$$\mathbf{S}_\gamma = \begin{cases} 1, & \text{if } j \in \mathcal{P}_{\text{out}} \text{ and } i \in \mathcal{P}_{\text{in}} \text{ and there is a feedback connection} \\ & \text{from } j \text{ to } i; \\ \mathbf{S}_{ij}, & \text{if } i \in \mathcal{P}_{\text{out}} \text{ and } j \in \mathcal{P}_{\text{in}} \text{ and both } i \text{ and } j \text{ are in the same plant} \\ & \text{and where } \mathbf{S}_{ij} \text{ is the scattering component from } j \text{ to } i; \\ 0, & \text{otherwise.} \end{cases}$$

More generally, if γ is an ordered sequence (paths) of ports, alternating between input and output ports, we define \mathbf{S}_γ inductively. If γ is the concatenation of γ_1 followed by γ_2 then

$$\mathbf{S}_\gamma = \mathbf{S}_{\gamma_2} \mathbf{S}_{\gamma_1}.$$

Finally, we set $\Gamma(i, j)$ to be the set of all paths going from j to i . Then for $i \in \mathcal{P}_{\text{out}}^{\text{ext}}, j \in \mathcal{P}_{\text{in}}^{\text{ext}}$

$$\begin{aligned}
(\mathbf{S}_{\text{net}})_{ij} &= \sum_{\gamma \in \Gamma(i, j)} \mathbf{S}_\gamma; \\
(\mathbf{L}_{\text{net}})_i &= \sum_{k \in \mathcal{P}_{\text{out}}} \sum_{\gamma \in \Gamma(i, k)} \mathbf{S}_\gamma \mathbf{L}_k; \\
\mathbf{H}_{\text{net}} &= \mathbf{H}_0 + \sum_{l, k \in \mathcal{P}_{\text{out}}} \sum_{\gamma \in \Gamma(l, k)} \mathbf{L}_l^\dagger \mathbf{S}_\gamma \mathbf{L}_k,
\end{aligned} \tag{17}$$

where \mathbf{H}_0 is the sum of the individual component systems Hamiltonians.

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References

- [1] L. Accardi, R.L. Hudson: *Quantum stochastic flows and non-abelian co-homology* Quantum Probability V, Lecture Notes in Mathematics **1442**, 54-69 (1990)
- [2] V. P. Belavkin: Optimization of Quantum Observation and Control, In Proc. of 9th IFIP Conf on Optimizat Techn. Notes in Control and Inform Sci **1**, Springer-Verlag, Warszawa (1979)
- [3] V. P. Belavkin: Theory of the Control of Observable Quantum Systems. Automatica and Remote Control **44** (2) 178–188 (1983).
- [4] V.P. Belavkin. Quantum continual measurements and a posteriori collapse on CCR. Commun. Math. Phys., **146**:611635, (1992)
- [5] V. P. Belavkin: On Quantum Ito Algebras and Their Decompositions. Letters in Mathematical Physics **45**, 131-145 (1998)
- [6] L. Bouten, R. Van Handel, and M.R. James. An introduction to quantum filtering. math.OC/06011741 (2006)
- [7] H.J. Carmichael. Quantum trajectory theory for cascaded open systems. Phys. Rev. Lett., **70**(15):22732276, (1993)
- [8] C.M. Caves, Quantum limits on noise in linear amplifiers, Phys. Rev. D., **26**, 1817, (1982)
- [9] A.M. Chebotarev, Quantum stochastic differential equation is unitarily equivalent to a symmetric boundary problem in Fock space, Inf. Dim. Anal. Quantum Prob., **1**, 175-199, (1998)
- [10] E.B. Davies, One-parameter Semigroups, Academic Press Inc, London (1980)
- [11] C.W. Gardiner. Driving a quantum system with the output field from another driven quantum system. Phys. Rev. Lett., **70**(15):22692272, 1993.
- [12] C.W. Gardiner and M.J. Collett. Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation. Phys. Rev. A, **31**(6):37613774, (1985)
- [13] J. Gough, Quantum flows as Markovian limit of emission, absorption and scattering interactions, Commun. Math. Phys. **254**, no. 2, 489-512 (2005)

- [14] J. Gough, V.P. Belavkin, O.G. Smolyanov, Hamilton-Jacobi-Bellman equations for Quantum Filtering and Control, *J. Opt. B: Quantum Semiclass. Opt.* **S237-244**, Special issue on quantum control, (2005)
- [15] J. Gough, M.R. James, The series product and its application to feed-forward and feedback networks, arXiv:07070048(v1) [quant-ph], to appear *IEEE Trans. Automatic Control*
- [16] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice Hall, New Jersey 07632, U.S.A. (1995)
- [17] M. Gregoratti, The Hamiltonian operator associated to some quantum stochastic differential equations, *Commun. Math. Phys.*, **222**, 181-200, (2001)
- [18] R.L. Hudson and K.R. Parthasarathy. Quantum Ito's formula and stochastic evolutions. *Commun. Math. Phys.*, 93:301323, (1984)
- [19] S. Lloyd. Coherent quantum feedback. *Phys. Rev. A*, 62:022108, (2000)
- [20] P. Warszawski, H.M. Wiseman, and H. Mabuchi. Quantum trajectories for realistic detection. *Phys. Rev. A*, 65:023802, (2002)
- [21] H. Wiseman. Quantum theory of continuous feedback. *Phys. Rev. A*, 49(3):21332150, (1994)
- [22] M. Yanagisawa and H. Kimura. Transfer function approach to quantum control-part I: Dynamics of quantum feedback systems. *IEEE Trans. Automatic Control*, (48):21072120, 12 (2003)
- [23] M. Yanagisawa and H. Kimura. Transfer function approach to quantum control-part II: Control concepts and applications. *IEEE Trans. Automatic Control*, (48):21212132, 12 (2003)
- [24] N. Young, *An Introduction to Hilbert Space*, Cambridge Mathematical Textbooks, (1988)
- [25] B. Yurke and J.S. Denker. Quantum network theory. *Phys. Rev. A*, **29**(3):14191437, (1984)
- [26] K. Zhou and J. Doyle and K. Glover. *Robust and Optimal Control*. Prentice Hall, NJ, 1996.